Integrals Involving Exp. & Log Functions

Rule:

① $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

② $\int b^{ax}\,dx = \frac{b^{ax}}{a\ln b} + c$

③ $\int x^{-1}\,dx = \int \frac{dx}{x} = \ln|x| + c$

④ $\int \ln x\,dx = x(\ln x - 1) + c$

⑤ $\int \log_b x\,dx = \frac{x}{\ln b}(\ln x - 1) + c$

Eg. 1. Find an Antiderivative of $e^x\sqrt{1+e^x}$

Solⁿ: For an antiderivative of $e^x\sqrt{1+e^x}$, we find the indefinite integral of $e^x\sqrt{1+e^x}$.

Now,

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{1+e^x}\cdot e^x dx$$

$u = 1+e^x$
$\Rightarrow du = e^x dx$

$$= \int \sqrt{u}\,du$$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{1/2+1}}{1/2+1} + C$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1+e^x)^{3/2} + C$$

Alternative Substitution:

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{1+e^x}\cdot e^x dx$$

$u = e^x$
$\Rightarrow du = e^x dx$

$$= \int \sqrt{1+u}\cdot du$$

$$= \int \sqrt{v}\, dv$$

$v = 1+u$
$\Rightarrow dv = (0+1)du$
$= du$

$$= \frac{2}{3} v^{3/2} + C$$

$$= \frac{2}{3}(1+u)^{3/2} + C$$

$$= \frac{2}{3}(1+e^x)^{3/2} + C$$

Eg. 2. Evaluate the indefinite integral $\int 3x^2 e^{2x^3} dx$

Sol^n:- Observation $\longrightarrow$ exponent is $(2x^3)$ & if we take the derivative of it we get $6x^2$, which kind of matches our problem.

Let $u = 2x^3$. Then $du = 2(3x^2)\,dx$
$= 6x^2 dx$

So. $$\int 3x^2 e^{2x^3} dx = \frac{1}{2}\int 2\left(3x^2 e^{2x^3}\right) dx$$

$$= \frac{1}{2}\int e^{2x^3}\left(6x^2 dx\right)$$

$$= \frac{1}{2}\int e^{u}\,du$$

$$= \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}e^{2x^3} + C$$

Eg 3. Find the price-demand equation for a toothpaste of some brand in Walmart, when the demand is 50 tubes/week & price is \$2.35/tube. It is given that the marginal price-demand function

$p'(x) = -0.015e^{-0.01x}$, where x is the number of tubes per week. [Given, $e^{-0.5} = 0.6$]

Now if Walmart sells 100 tubes/week, then what will be the profitable price for each tube.

Solⁿ:- To find the price-demand equation, we need to integrate the marginal price-demand function.

Thus, $p(x) = \int p'(x)\,dx$

$$= \int(-0.015e^{-0.01x})\,dx$$

$$= -0.015 \int e^{-0.01x}\,dx$$

$$= -0.015 \cdot \frac{e^{-0.01x}}{(-0.01)} + C$$

$$= 1.5\, e^{-0.01x} + C$$

Now we know that, when the price was \$2.35/tube the demand is 50 tube/week.

ie, $p(50) = 2.35$

But $p(50) = 1.5\, e^{-0.01(50)} + C$

$\Rightarrow 2.35 = 1.5\, e^{-0.5} + C$

$\Rightarrow C = 2.35 - 1.5 e^{-0.5}$

$= 2.35 - 1.5\,(0.6)$

$= 2.35 - 0.9$

$= 1.45$

Hence, $p(x) = 1.5 e^{-0.01x} + 1.45$

For the 2nd part, we need to find $p(100)$.

Since we know the price-demand equation, we just need to plug in $x = 100$ in $p(x)$.

ie, $p(100) = 1.5e^{-0.01(100)} + 1.45$

$= 1.5\, e^{-1} + 1.45$

$\approx 2.00.$

So, the profitable selling price for a tube will be \$2.00 if Walmart is selling 100 tubes/week.

Eg. 4 Find the antiderivative of $\frac{1}{x+2}$, $x \neq -2$

Soln: Note: $\frac{1}{x+2} = (x+2)^{-1}$ & if we choose $u = x+2$, then we can use the rule for u^{-1}.

Now, $du = \left(\frac{du}{dx}\right) dx$

$= (1+0)\, dx$

$= dx$

So, $\int \frac{1}{x+2}\, dx = \int \frac{1}{u}\, du = \ln|u| + C$

$= \ln|x+2| + C$